Spreading for the generalized nonlinear Schrödinger equation with disorder

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The dynamics of an initially localized wave packet is studied for the generalized nonlinear Schrödinger equation with a random potential, where the nonlinear term is $\beta|\psi|^p\psi$ and p is arbitrary. Mainly short times for which the numerical calculations can be performed accurately are considered. Long time calculations are presented as well. In particular, the subdiffusive behavior where the average second moment of the wave packet is of the form $\langle m_2\rangle \approx t^\alpha$ is computed. Contrary to former heuristic arguments, no evidence for any critical behavior as function of p is found. The properties of $\alpha(p)$ for relatively short times are explored, a scaling property and a maximal value for $p\approx\frac{1}{2}$ are found.

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We consider the discrete nonlinear Schrödinger equation (NLSE) with a random potential in one dimension:

$$i\frac{\partial\psi_n}{\partial t} = -\psi_{n+1} - \psi_{n-1} + \varepsilon_n\psi_n + \beta|\psi_n|^p\psi_n, \qquad (1)$$

where ε_n are independent and identically distributed random variables uniformly distributed in the interval $[-\frac{W}{2},\frac{W}{2}]$, p is the degree of nonlinearity, and β is its strength. For $\beta=0$ this equation reduces to the Anderson model where all the states are exponentially localized [1]. Consequently, for $\beta=0$ if one starts with a localized wave packet it will not spread indefinitely. In the absence of a random potential spreading takes place for all p [2]. In fact the continuous version of Eq. (1) for $p=2$ and without the disorder is integrable [2].

The case of $p=2$ is of experimental relevance in classical optics [3] and in the field of Bose-Einstein condensates (BECS), where the NLSE is known as the Gross-Pitaevskii equation [4,5]. Recent experiments in this field include spreading of light waves in two-dimensional disordered lattices [3] and one-dimensional wave guides [6], which were found experimentally to exhibit localization for sufficiently strong disorder. In other experiments [7] matter waves propagating through continuous-wave guides formed by optical potentials and localization is found for long wavelengths (when the potential in the wave guide can be considered as random). The transport properties of a BEC which is incident on some finite disordered region were explored theoretically [8]. Because of the nonlinear nature of the problem it is not simply related to the present work. The transport regimes of the scattered BEC may vary from superfluidity to Anderson localization, and the interactions between the atoms play a major role in the dynamics [8]. In related work [9], there is a suggestion for an experimental realization where the potential energy is due to interaction between the magnetic moment of the atoms and a fluctuating magnetic field which is generated by an inhomogeneous wire.

The NLSE was studied extensively in the recent years, mainly for $p=2$. In particular, the growth of the second moment was explored, and it was found (numerically) to grow subdiffusively [10–12], namely, for a particle initially at $n=0$,

$$\langle m_2(t)\rangle = Dt^\alpha, \qquad (2)$$

where $m_2=\Sigma_n n^2|\psi_n|^2$ and α was found asymptotically in time to be $\alpha\approx 0.33$ (for $p=2$). The average $\langle\ldots\rangle$ is an average over the realizations of the random potential.

The analytical and intuitive understanding of Eq. (1) is quite poor. The simplest intuitive argument suggests that if a wave packet spreads for a long enough time, the amplitude of the wave packet becomes negligible (since the norm, $\Sigma_n|\psi_n|^2=1$, is conserved) and as a result the nonlinear term weakens and becomes irrelevant. Consequently, localization takes place. The difficulty with this argument (in addition with the fact that it disagrees with numerical results [10–12]) is that although the absolute value of the nonlinear term becomes smaller it should be compared to an energy scale that may decrease as well. For $p=2$ such an argument was developed by Pikovsky and Shepelyansky [10]. (It is very similar to an argument that was found to work remarkably well for another system [13].) We generalize this argument to an arbitrary value of p. Assuming that after some time the packet ψ is spread over Δn sites while its norm is preserved than that typically $|\psi_n|^2\approx\frac{1}{\Delta n}$ and therefore the nonlinear term produces an energy shift of the order $\delta E=\beta|\psi_n|^p$ that is of the order of $\delta E\approx\Delta n^{-p/2}$. Comparing this term with the typical distance between the energies of the linear problem, $\Delta E\approx\frac{1}{\Delta n}$, gives $\frac{\delta E}{\Delta E}\approx\beta\Delta n^{-(p-2)/2}$. Based on this argument, Pikovsky and Shepelyansky that were interested in the case $p=2$, where $\frac{\delta E}{\Delta E}\approx\beta$ concluded that there is a critical value denoted by β_c such that for $\beta<\beta_c$ the nonlinear term is negligible compared to the level spacings of the linear problem and therefore Anderson localization holds. For $\beta>\beta_c$ the states of the linear problem are mixed and presumably Anderson localization breaks down and spreading takes place. From this argument it turns out that $p=2$ is a critical degree of nonlinearity, and for $p>2$, $\frac{\delta E}{\Delta E}\rightarrow 0$ as Δn grows and localization holds. Existence of a critical value of p was not considered in [10] since only the case $p=2$ was studied. Also for the nonlinear Schrödinger equation without disorder, $p=2$ has a critical meaning [14,15]. In the present Brief Report *no evidence for the criticality at* $p=2$ was found. This leads one to question the validity of the arguments implying the criticality of $p=2$ for spreading. Recently, Flach and coworkers presented arguments that in the long time limit α

$=\frac{1}{p+1}$ and there is no critical value of β or p [11,12,16]. Their arguments are supported by some numerical calculations. It is unclear how α should behave when the limit $p\to 0$ is taken since in this limit localization takes place and one expects $\alpha=0$. This is another motivation for the present work. Some arguments presented in [10–12] involve assumptions on chaoticity of various modes. The present work does not test these assumptions.

There are conjectures based on a perturbation theory [17] and rigorous results [18] claiming that asymptotically the second moment of the wave packet cannot grow faster than logarithmically as a function of time. Nevertheless, numerical data predict a power-law growth of the second moment. If we trust the conjectures (their violation will be very surprising and of great interest), it is reasonable that the available numerical data are either not asymptotic (the time scale of this problem is unknown and therefore also the time when the system enters the asymptotic regime) or not reliable due to computational errors. Considering this, we concentrate on the short time behavior of a wave packet. Some results for the long time behavior are also presented for completeness.

In order to follow the dynamics of a wave packet, we use the SABA algorithm (see Ref. [12]), which belongs to the family of split step algorithms and evaluates the wave packet in small steps, changing from coordinate space to momentum space. We apply the disorder and nonlinear interactions in the coordinate space, transform the wave to momentum space and apply there the kinetic-energy term, transform it back to the coordinate space, and so on. Nearly all numerical calculations for this problem use such methods. Additional details on the SABA algorithm can be found in [12]. Like any numerical algorithm, the SABA algorithm accumulates errors during the calculation which grow with the time of the integration. We use two criteria to determine whether our results are reliable or not: (**t1**) time reversal and (**t2**) comparison with data which are obtained using smaller time steps. Time reversal means integrating Eq. (1) from time 0 to some later time and then integrating back to time 0. At the end of this process (if there are no errors) we should get the initial wave packet. To measure the accumulated errors, we define $\delta_{tr}=\Sigma_n|\psi_{initial}-\psi_{reversed}|$ and demand $\delta_{tr}<0.1$. The comparison with smaller time step is done as follows: we calculate the second moment $m_2(t)$ for representative realizations and then recalculate it using a smaller time step (half of the original one). We define

$$\delta_{m_2}=\frac{1}{T}\int\left|\frac{m_{2,\delta t}-m_{2,1/2\delta t}}{m_{2,1/2\delta t}}\right|dt \tag{3}$$

and demand $\delta_{m_2}<0.01$. Nearly all published numerical calculations used a more relaxed test: (**t3**) where in (**t2**) m_2 is replaced by the average over realizations.

We calculate m_2 for various values of β and p and average over 5000 realizations until time 1000. We verified that (**t1**) and (**t2**) are satisfied for representative realizations. We use time steps of duration 0.1, 0.02, 0.01, and 0.000 25 for $\beta=0.25$, 0.5, 0.75, and 1, respectively, that were required to satisfy (**t1**) and (**t2**). In addition, data are presented for $\beta=2$ and 4 using time steps of 0.1 where (**t1**) and (**t2**) are not

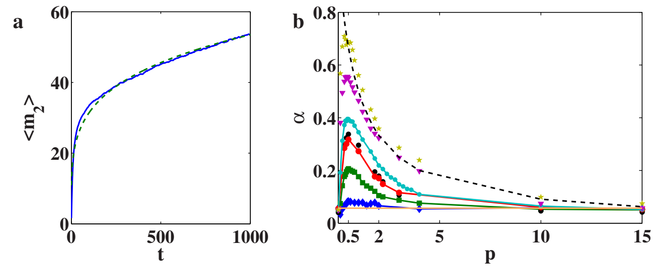

FIG. 1. (Color online) (a) $\langle m_2\rangle$ for $\beta=1$ and $p=2$ as a function of time. The blue solid curve is the second moment calculated numerically and the green dashed line is the fit which we use in order to find α. (b) $\alpha(p)$ for different values of β. From top to the bottom: $\beta=4,2,1,0.75,0.5,0.25$ (yellow stars, purple triangles, turquoise asterisks, red or black circles, green squares, and blue diamonds). For $\beta=0.75$ we present two lines: the solid red line and circles are based on $\langle m_2\rangle$ and the black circles are based on $\langle \ln m_2\rangle$. Only the data for $\beta=0.25$, 0.5, 0.75, and 1 where points are connected by lines satisfy (**t1**) and (**t2**). The black dashed line is the asymptotic prediction $\alpha=\frac{1}{1+p}$ for long times [12]. The linear case ($\beta=0$) is represented by orange solid line for comparison. For all realizations, $W=4$ and maximal localization length is of 6 lattice sites. At the initial time the wave packet populates one site ($n=0$).

satisfied. The results are shown in Fig. 1(b) where α is obtained from fits similar to the one presented in Fig. 1(a). Only the data in the interval $500\le t\le 1000$ that does not involve the initial spread were used in the fit of α. If we choose the time interval to be $300\le t\le 1000$ or $800\le t\le 1000$, our results do not change in a significant way. If we choose to calculate $\langle \ln m_2\rangle$ instead of $\langle m_2\rangle$, it does not affect the results [as demonstrated in Fig. 1(b) for $\beta=0.75$]. As we could expect, $\alpha(p\to 0)\to 0$, and when p is large, α is very small. The maximal α is obtained for $p\approx\frac{1}{2}$ and nothing special happens for $p=2$. We do not see any discontinuity for $p=0$. All the lines in Fig. 1(b) have similar shape. After forming linear transformations

$$\bar{\alpha}=c_1\alpha+c_2, \tag{4}$$

where c_1 and c_2 are some numerical constants that are independent of α and p and depend only on β, all the lines approximately coincide as shown in Fig. 2, leading us to the

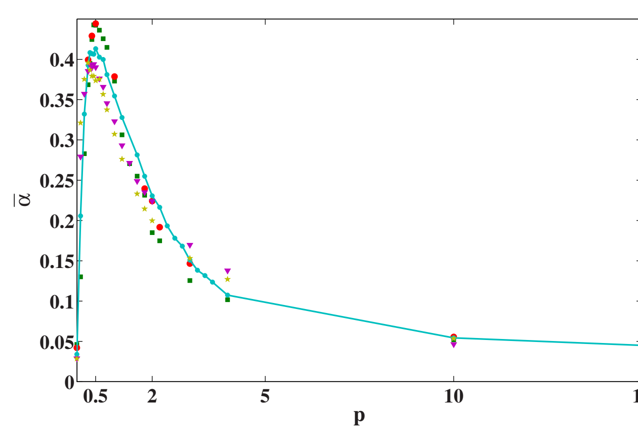

FIG. 2. (Color online) Figure 1(b) after rescaling by Eq. (4).

conclusion that there might be some scaling property. We were unable to find the significance of the constants c_1 and c_2 [19]. In the present work, the values of α were calculated for short times where the numerical errors can be well controlled but long enough to see the growth of the second moment in the presence of the nonlinear term (contrary to the linear case). It is used just as a measure of the dependence on p and it is not directly related to the asymptotic exponents calculated in [10–12,16].

In our short time runs ($t \le 1000$) the wave packet did not spread over many sites. In the case of maximal spreading ($\beta=4$, $p=\frac{1}{2}$), the second moment reached to a maximal value of 150, and for the parameters $\beta=1$, $p=2$ the second moment was slightly smaller than 55, while the localization length is about 6. When we follow the dynamics for longer times [for which (**t1**) and (**t2**) are not satisfied but (**t3**) is satisfied] the results support our previous conclusion that nothing critical happens for $p=2$. We see that the wave packet spreads for all powers of nonlinearity p in a similar way, as shown in Fig. 3 for $p=0$, 1.5, 2, 2.5, 4, and 8. Similar results are found in detailed studies of Mulansky [20].

In conclusion, we have found that for short times, there is no evidence of any critical phenomena neither for $p=0$ nor for $p=2$. This conclusion is supported by long time calculations. In addition, we found that α that is found for the short time regime $t \le 1000$ has a maximum for $p \approx \frac{1}{2}$, and there is evidence for scaling (Fig. 2). Understanding the physics of the $\alpha(p)$ plots, explaining why is the maximal spreading obtained for $p \approx \frac{1}{2}$, explaining of the origin of the scaling, and finding the asymptotic long time behavior of $\alpha(p)$ are left for future research.

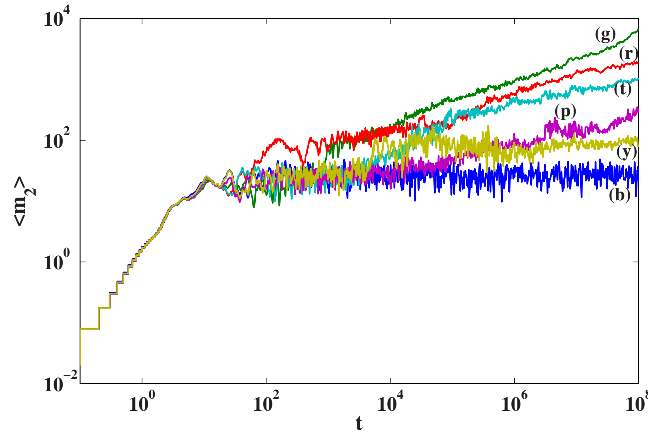

FIG. 3. (Color online) $m_2(t)$ for a representative realization as a function of time for $\beta=1$ and $W=4$. From top to bottom: $p=1.5, 2, 2.5, 4, 8, 0$, (green, red, turquoise, purple, yellow, and blue).

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